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Entropy and entanglement dynamics in a quantum deformed coupled system

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Abstract
A class of supersymmetric bound-state problems which represent the coupling of a two-level atom or molecule with a quantum-deformed shape-invariant potential system is introduced. We study the quantum dynamics of the partial entropies and the entanglement of the bipartite system as well as the excitation of the two-level atom for two possible forms of the pure initial state of the coupled system.

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1. Introduction

Recognized as one of the most striking features of quantum mechanics, the entanglement phenomenon has been extensively studied in recent years. This interest is due to: (i) its applications in the quantum information processing such as quantum computing [1], teleportation [2], cryptography [3], dense coding [4] and entanglement swapping [5]; (ii) its potential ability to give new insights into understanding many physical phenomena including super-radiance [6], superconductivity [7], disordered systems [8], etc. Because of its sensitivity to all moments of the density operator, one very useful operational tool to quantify the entanglement of a given quantum systems is the entropy. The concept of entropy, coming from thermodynamics, has been reconsidered recently in the context of quantum information theory because of its connection to the entanglement of the system.

On the other hand, the development of quantum groups and quantum algebras motived great interest in q-deformed algebraic structures. Quantum groups and algebras have their origin in the quantum inverse problem method [9] and the first structure appeared in the studies of solutions to the Yang–Baxter equation [10]. The quantum Yang–Baxter equation is by now known to play a profound role in a variety of diverse problems in theoretical physics [11, 12]. Quantum algebras are deformed versions of the usual Lie algebras obtained by introducing a deformation parameter q. In this sense, the quantum algebras provide us with a class of symmetries which is richer than the usual Lie symmetries; the latter is contained in the former as a special case (when $q \to 1$). Until now quantum groups have found applications in solid-state physics [13], nuclear physics [14, 15], quantum optics [16], conformal field theories [17] and more formal mathematical problems.

We know that fully solvable models in quantum theory are so rare that they are worth studying in their own right. Although they are in general oversimplified, their solutions give a clear understanding of the physical phenomena studied and, as a first approach, make it possible to control some approximations indispensable for the treatment of more realistic models. The coupled-channel models based on the dipole and rotating wave approximations are examples of such a restricted set of solvable models. The simplicity, analyticity and strong correlation which develop dynamically among the subsystems during their interaction make these coupled-channel models excellent laboratories for investigating the quantum entanglement. In spite of their apparent simplicity, they exhibit a quite complicated behaviour and fully quantum-mechanical effects that are strongly dependent on the initial conditions of the system. The Jaynes–Cummings model [18], extensively used with success in quantum optics [19] in the description of a single two-level atom or molecule resonantly coupled to a single mode of the quantized electromagnetic field is the most known example of this sort.

In this paper we introduce a class of supersymmetric coupled-channel problems based on the dipole and rotating wave approximations, consisting of a quantum deformed and shape-invariant system [20] interacting with a two-level atom or a molecule. This is a non-trivial coupled-channel problem which may find applications in molecular, atomic and nuclear physics. We introduce the q-deformed supersymmetric and shape-invariant Hamiltonian of the coupled-channel system and evaluate the dynamical evolution of the system and partial entropies of the subsystems in terms of the deformation parameter q. We consider two possible forms for the initial quantum state of the q-deformed shape-invariant potential system: a purely coherent and a purely squeezed states.

For the sake of completeness we will briefly present the fundamental principles of the algebraic formulation to shape invariance and the basic facts of the algebraic quantum deformed shape-invariant model in section 2. In section 3, we introduce the Hamiltonian of the coupled system and obtain its time evolution operator and the density operator; in section 4 we obtain the temporal behavior of the population inversion factor and the coupling potential partial entropy; in section 5 we apply our generalized results for a quantum deformed Pöschl–Teller potential system, discussing the relevant aspects of the time behavior of each observable as a function of the deformation parameter q. Conclusions are given in section 6.

2. Algebraic formulation for quantum deformation shape-invariant systems

Recently, the use of the operator techniques based on algebraic models [21–24] brought renewed interest to the study of shape-invariant systems. Coupled with the supersymmetry concept, the algebraic formulation of the shape invariance has proved to be a powerful and elegant quantum technique to exactly solve a set of potential systems with applications in molecular, atomic and nuclear physics. These techniques deal with one-dimensional partner Hamiltonians $\hat{H}_\pm$, written in terms of the operator $\hat{A}(a_1) \equiv \frac{1}{\sqrt{\hbar\Omega}}\left\{W(a_1;x) + \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}_x\right\}$ and its adjoint operator as

$$\begin{aligned}\hat{H}_- &= \frac{\hat{p}_x^2}{2M} + V_-(a_1;x) = \hbar\Omega\hat{A}^\dagger(a_1)\hat{A}(a_1) \qquad \text{and}\\ \hat{H}_+ &= \frac{\hat{p}_x^2}{2M} + V_+(a_1;x) = \hbar\Omega\hat{A}(a_1)\hat{A}^\dagger(a_1)\end{aligned} \tag{1}$$

where a_1 is a set of potential parameters, $\hbar\Omega$ is a constant energy scale factor and the superpotential $W(a_1; x)$ is a real function related to the partner potentials via $V_{\pm}(a_1; x) = W^2(a_1; x) \pm \frac{\hbar}{\sqrt{2M}} \mathrm{d}W(a_1; x)/\mathrm{d}x$. Introducing the parameter translation operator $\hat{T}$ and the similarity transformation $\hat{T}\hat{O}(a_1)\hat{T}^{\dagger} = \hat{O}(a_2)$ that replaces a_1 with a_2 in a given operator or function and the operators [21]

$$\hat{B}_+ = \hat{A}^{\dagger}(a_1)\hat{T} \qquad \text{and} \qquad \hat{B}_- = \hat{B}_+^{\dagger} = \hat{T}^{\dagger}\hat{A}(a_1) \tag{2}$$

the partner Hamiltonians in (1) take the forms $\hat{H}_- = \hbar\Omega\hat{\mathcal{H}}_-$ and $\hat{H}_+ = \hbar\Omega\hat{T}\hat{\mathcal{H}}_+\hat{T}^{\dagger}$, where $\hat{\mathcal{H}}_{\pm} = \hat{B}_{\mp}\hat{B}_{\pm}$. With these definitions, the condition of shape invariance $\hat{A}(a_1)\hat{A}^{\dagger}(a_1) = \hat{A}^{\dagger}(a_2)\hat{A}(a_2) + R(a_1)$, first obtained in [25], can be written as the commutation relation

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^{\dagger}R(a_1)\hat{T} \equiv R(a_0). \tag{3}$$

In the cases studied so far, the parameters a_1 and a_2 are related by either a translation [21, 26] or a scaling [24, 27, 28]. Since the ground state of the Hamiltonian $\hat{\mathcal{H}}_-$ satisfies the annihilation condition $\hat{A}|0\rangle = 0 = \hat{B}_-|0\rangle$, then using the additional relations $\hat{B}_{\pm}R(a_n) = R(a_{n\pm1})\hat{B}_{\pm}$ it is possible to obtain the eigenvalue equation $\hat{\mathcal{H}}_-|n\rangle = e_n|n\rangle$ and $\hat{\mathcal{H}}_+|n\rangle = \{e_n + R(a_0)\}\,|n\rangle$ where the normalized nth excited eigenstate $|n\rangle = \hat{K}_+^n|0\rangle$ can be obtained from the ground state by the action of the raising operator $\hat{K}_+ \equiv \frac{1}{\sqrt{\hat{\mathcal{H}}_-}}\hat{B}_+$ and the related eigenvalues are given by

$$e_0 = 0 \qquad \text{and} \qquad e_n = \sum_{k=1}^{n} R(a_k), \qquad \text{for} \quad n \geqslant 1. \tag{4}$$

With the above results above it is possible to show that

$$\hat{B}_+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle \qquad \text{and} \qquad \hat{B}_-|n\rangle = \sqrt{e_{n-1} + R(a_0)}|n-1\rangle, \tag{5}$$

making clear the ladder nature of the operators $\hat{B}_{\pm}$ when applied on the eigenstates $\{|n\rangle; n = 0, 1, 2, \ldots\}$ of $\hat{\mathcal{H}}_-$.

The study of quantum deformed systems other than the harmonic oscillator is very recent and presents new and interesting aspects. There is an alternative quantum deformed model developed for shape-invariant systems which, unlike the others, preserve the shape invariance condition after the deformation process [29]. The results obtained with this model can be summarized by introducing the real parameter q $(0 < q \leqslant 1)$ and defining the quantum deformed ladder operators

$$\begin{aligned} \hat{S}_-^{(q)} &= \mathcal{F}_q q^{\frac{1}{2}\hat{\mathcal{H}}_+}\hat{B}_-^{(q)} = \mathcal{F}_q\hat{B}_-^{(q)}q^{\frac{1}{2}\hat{\mathcal{H}}_-} \qquad \text{and} \\ \hat{S}_+^{(q)} &= \hat{S}_-^{(q)\dagger} = q^{\frac{1}{2}\hat{\mathcal{H}}_-}\hat{B}_+^{(q)}\mathcal{F}_q = \hat{B}_+^{(q)}q^{\frac{1}{2}\hat{\mathcal{H}}_+}\mathcal{F}_q \end{aligned} \tag{6}$$

where $\mathcal{F}_q$ is a compact notation for a real functional of $(a_0, a_1, a_2, \ldots)$. The standard q-deformed form for $\hat{B}_{\pm}$

$$\hat{B}_{\pm}^{(q)} \equiv \left\{\hat{B}_{\mp}^{(q)}\right\}^{\dagger} = \hat{B}_{\pm}\sqrt{\frac{[\hat{\mathcal{H}}_{\pm}]_q}{\hat{\mathcal{H}}_{\pm}}} = \sqrt{\frac{[\hat{\mathcal{H}}_{\mp}]_q}{\hat{\mathcal{H}}_{\mp}}}\hat{B}_{\pm} \tag{7}$$

is defined with the q-operators extension of the q-number definition $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$ and the property $\hat{B}_{\pm}f(\hat{\mathcal{H}}_{\pm}) = f(\hat{\mathcal{H}}_{\mp})\hat{B}_{\pm}$, valid for any analytical function $f(x)$. Assuming that the functional $\mathcal{F}_q$ satisfies the constraint

$$\hat{T}\mathcal{F}_q\hat{T}^{\dagger} = q^{R(a_0)}\mathcal{F}_q \tag{8}$$

and taking into account the operator relation $(q - q^{-1})q^{\hat{\mathcal{H}}_{\mp}}[\hat{\mathcal{H}}_{\pm}]_q = q^{(\hat{\mathcal{H}}_+ + \hat{\mathcal{H}}_-)} - q^{\mp R(a_0)}$ we can obtain the commutator

$$\left[\hat{S}_-^{(q)}, \hat{S}_+^{(q)}\right] = \mathcal{G}_0^{(q)} \qquad \text{where} \qquad \mathcal{G}_0^{(q)} \equiv \mathcal{F}_q^2 q^{R(a_0)}[R(a_0)]_q. \tag{9}$$

Comparing equations (3) and (9) we conclude that the latter can be associated with a shape-invariance condition as the former with the remainder $R(a_0)$ replaced by $\mathcal{G}_0^{(q)}$. Commutator (9) suggests that $\hat{S}_-^{(q)}$ and $\hat{S}_+^{(q)}$ are the appropriate creation and annihilation operators for the spectra of the q-deformed shape-invariant systems whose partner Hamiltonians are defined as

$$\hat{\mathcal{H}}_-^{(q)} \equiv \hat{S}_+^{(q)}\hat{S}_-^{(q)} = q^{2R(a_0)}\mathcal{F}_q^2 q^{\hat{\mathcal{H}}_-}[\hat{\mathcal{H}}_-]_q \qquad \text{and} \qquad \hat{\mathcal{H}}_+^{(q)} \equiv \hat{S}_-^{(q)}\hat{S}_+^{(q)} = \mathcal{F}_q^2 q^{\hat{\mathcal{H}}_+}[\hat{\mathcal{H}}_+]_q. \quad (10)$$

Taking into account that $[\hat{\mathcal{H}}_-, \mathcal{G}_0^{(q)}] = 0 = [\hat{\mathcal{H}}_-, \mathcal{F}_q]$ and expression (10) we can show that $[\hat{\mathcal{H}}_-, \hat{\mathcal{H}}_-^{(q)}] = 0$ and thus these Hamiltonians have the common set of eigenstates $\{|n\rangle; n = 0, 1, 2, \ldots\}$. The eigenvalues of $\hat{\mathcal{H}}_-^{(q)}$ can be obtained by using definitions (6), (7), (10) and equation (9) to write the additional commutation relations

$$\begin{aligned} \left[\hat{\mathcal{H}}_-^{(q)}, \left(\hat{S}_+^{(q)}\right)^n\right] &= +\left\{\mathcal{G}_1^{(q)} + \mathcal{G}_2^{(q)} + \cdots + \mathcal{G}_n^{(q)}\right\}\left(\hat{S}_+^{(q)}\right)^n, \\ \left[\hat{\mathcal{H}}_-^{(q)}, \left(\hat{S}_-^{(q)}\right)^n\right] &= -\left(\hat{S}_-^{(q)}\right)^n\left\{\mathcal{G}_1^{(q)} + \mathcal{G}_2^{(q)} + \cdots + \mathcal{G}_n^{(q)}\right\} \end{aligned} \quad (11)$$

where $\mathcal{G}_k^{(q)} = \hat{T}\mathcal{G}_{k-1}^{(q)}\hat{T}^\dagger$. From the ground-state annihilation condition $\hat{S}_-^{(q)}|0\rangle = 0$ and commutators (9) and (11) it follows that

$$\hat{\mathcal{H}}_-^{(q)}\left\{\left(\hat{S}_+^{(q)}\right)^n|0\rangle\right\} = \left\{\mathcal{G}_1^{(q)} + \mathcal{G}_2^{(q)} + \cdots + \mathcal{G}_n^{(q)}\right\}\left\{\left(\hat{S}_+^{(q)}\right)^n|0\rangle\right\} = \mathcal{E}_n^{(q)}\left\{\left(\hat{S}_+^{(q)}\right)^n|0\rangle\right\} \quad (12)$$

$$\hat{\mathcal{H}}_+^{(q)}\left\{\left(\hat{S}_+^{(q)}\right)^n|0\rangle\right\} = \left\{\mathcal{G}_0^{(q)} + \mathcal{G}_1^{(q)} + \cdots + \mathcal{G}_n^{(q)}\right\}\left\{\left(\hat{S}_+^{(q)}\right)^n|0\rangle\right\} = \mathcal{A}_n^{(q)}\left\{\left(\hat{S}_+^{(q)}\right)^n|0\rangle\right\} \quad (13)$$

i.e., $|n\rangle \propto \left(\hat{S}_+^{(q)}\right)^n|0\rangle$ is an eigenstate of the Hamiltonians $\hat{\mathcal{H}}_\mp^{(q)}$ with the eigenvalues

$$\mathcal{E}_n^{(q)} = \sum_{k=1}^n \mathcal{G}_k^{(q)} = \sum_{k=1}^n \hat{T}^k\mathcal{F}_q^2\hat{T}^{\dagger k}q^{R(a_k)}\left[R(a_k)\right]_q = \mathcal{F}_q^2 q^{\{e_n+2R(a_0)\}}[e_n]_q \quad (14)$$

$$\mathcal{A}_n^{(q)} = \sum_{k=0}^n \mathcal{G}_k^{(q)} = \sum_{k=0}^n \hat{T}^k\mathcal{F}_q^2\hat{T}^{\dagger k}q^{R(a_k)}\left[R(a_k)\right]_q = \mathcal{F}_q^2 q^{\{e_n+R(a_0)\}}[e_n + R(a_0)]_q \quad (15)$$

where, to get these final expressions, we used the generalization of the condition in equation (8)

$$\hat{T}^k\mathcal{F}_q\hat{T}^{\dagger k} = \prod_{j=0}^{k-1} q^{R(a_j)}\mathcal{F}_q. \quad (16)$$

Note that, via the identity $R(a_n) = \hat{T}R(a_{n-1})\hat{T}^\dagger$ and condition (8), the two eigenvalues are related by $\mathcal{E}_n^{(q)} = \hat{T}\mathcal{A}_{n-1}^{(q)}\hat{T}^\dagger$. The ladder character of the operators $\hat{S}_\pm^{(q)}$ when acting on $|n\rangle$ follows from the relations

$$\hat{S}_+^{(q)}|n\rangle = \sqrt{\mathcal{E}_{n+1}^{(q)}}|n+1\rangle \qquad \text{and} \qquad \hat{S}_-^{(q)}|n\rangle = \sqrt{\mathcal{A}_{n-1}^{(q)}}|n-1\rangle, \quad (17)$$

obtained from the results above. In addition to the commutators in (11), we can establish the commutation relations

$$\left[\hat{S}_+^{(q)}, \mathcal{G}_j^{(q)}\right] = \left\{\mathcal{G}_{j+1}^{(q)} - \mathcal{G}_j^{(q)}\right\}\hat{S}_+^{(q)}, \qquad \left[\hat{S}_+^{(q)}, \left[\hat{S}_+^{(q)}, \mathcal{G}_j^{(q)}\right]\right] = \left\{\mathcal{G}_{j+2}^{(q)} - 2\mathcal{G}_{j+1}^{(q)} + \mathcal{G}_j^{(q)}\right\}\left(\hat{S}_+^{(q)}\right)^2, \quad (18)$$

and, in general, for the commutator of nth order we have

$$\underbrace{\left[\hat{S}_+^{(q)}, \left[\hat{S}_+^{(q)}, \left[\hat{S}_+^{(q)}, \ldots, \left[\hat{S}_+^{(q)}, \left[\hat{S}_+^{(q)}, \mathcal{G}_j^{(q)}\right]\right]\cdots\right]\right]\right]}_{\text{sequence of } n \text{ commutation operations}} = \left\{\sum_{k=0}^n (-1)^k \binom{n}{k}\mathcal{G}_{j+n-k}^{(q)}\right\}\left(\hat{S}_+^{(q)}\right)^n \quad (19)$$

which, with their adjoint commutation relations and equation (9), form an infinite-dimensional Lie algebra, realized here in a unitary representation.

For most shape-invariant systems we found that [29] $\lim_{q\to 1}\mathcal{F}_q = 1$. Thus taking into account the q-number definition and its limit when $q \to 1$, it is straightforward to show that $\lim_{q\to 1}\hat{S}_{\pm}^{(q)} = \hat{B}_{\pm}$ and the whole q-deformed algebraic formalism presented here reduces to that developed for the operators $\hat{B}_{\pm}$ [21].

3. A two-level atom coupled to a quantum deformed shape-invariant potential system

3.1. Hamiltonian

We consider in this study two interacting systems consisting of a single two-level atom or molecule coupled with a quantum deformed shape-invariant system which is associated with the ladder operators $\hat{S}_{+}^{(q)}$ and $\hat{S}_{-}^{(q)}$. The total Hamiltonian describing this coupled system may be written as $\hat{H}_{\mathrm{T}}^{(q)} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{P}}^{(q)} + \hat{H}_{\xi}^{(q)}$, where $\hat{H}_{\mathrm{A}}$ is the free atom Hamiltonian, $\hat{H}_{\mathrm{P}}^{(q)}$ is the Hamiltonian related to the quantum deformed shape-invariant potential system and $\hat{H}_{\xi}^{(q)}$ is the atom-potential interaction Hamiltonian. In treating a two-level system with a lower state $|-\rangle$ and an upper state $|+\rangle$ we can introduce the excitation $\hat{\sigma}_{+} \equiv |+\rangle\langle-|$ and de-excitation $\hat{\sigma}_{-} \equiv |-\rangle\langle+|$ operators as well as the inversion operator $\hat{\sigma}_3 \equiv |+\rangle\langle+|-|-\rangle\langle-|$ which satisfy the commutation relations $[\hat{\sigma}_{+}, \hat{\sigma}_{-}] = \hat{\sigma}_3$ and $[\hat{\sigma}_3, \hat{\sigma}_{\pm}] = \pm 2\hat{\sigma}_{\pm}$. By assuming a two-dimensional spinor representation for the eigenstates of the atomic system

$$\chi_{-} \equiv \langle\chi|-\rangle = \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad \chi_{+} \equiv \langle\chi|+\rangle = \begin{bmatrix}1\\0\end{bmatrix} \tag{20}$$

it is straightforward to verify that these operators will be represented by the 2×2 matrices

$$\begin{aligned}&\hat{\boldsymbol{\sigma}}_{+} = \chi_{+}\chi_{-}^{\dagger} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \qquad \hat{\boldsymbol{\sigma}}_{-} = \chi_{-}\chi_{+}^{\dagger} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \qquad \text{and}\\ &\hat{\boldsymbol{\sigma}}_{3} = \chi_{+}\chi_{+}^{\dagger} - \chi_{-}\chi_{-}^{\dagger} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.\end{aligned} \tag{21}$$

Thus if we define the polarization matrices $\hat{\boldsymbol{\sigma}}_1 \equiv \hat{\boldsymbol{\sigma}}_{-} + \hat{\boldsymbol{\sigma}}_{+}$ and $\hat{\boldsymbol{\sigma}}_2 \equiv \mathrm{i}(\hat{\boldsymbol{\sigma}}_{-} - \hat{\boldsymbol{\sigma}}_{+})$ we obtain the Pauli matrices $\hat{\boldsymbol{\sigma}}_i$, for $i = 1, 2$ and 3. If we consider that the eigenstates $|\pm\rangle$ of the Hamiltonian $\hat{H}_{\mathrm{A}}$ of a non-interacting two-level atom form a normalized basis then we must have $\langle\pm|\pm\rangle = 1$, $\langle\pm|\mp\rangle = 0$ and $|+\rangle\langle+| + |-\rangle\langle-| = \hat{\mathbb{1}}$. Therefore, using these relations together with the eigenvalue equation $\hat{H}_{\mathrm{A}}|\pm\rangle = \hbar\omega_{\pm}|\pm\rangle$ it is possible to write the free atom Hamiltonian in the form

$$\hat{H}_{\mathrm{A}} = \hat{\mathbb{1}}\hat{H}_{\mathrm{A}}\hat{\mathbb{1}} = \hbar\,(\omega_{+}\hat{\sigma}_{+}\hat{\sigma}_{-} + \omega_{-}\hat{\sigma}_{-}\hat{\sigma}_{+}) \tag{22}$$

where the projection operators $\hat{\pi}_{\pm} \equiv |\pm\rangle\langle\pm| = \hat{\sigma}_{\pm}\hat{\sigma}_{\mp}$ describe the population of the levels $\pm$ whose energies are $\hbar\omega_{\pm}$.

The Hamiltonian related to the quantum-deformed shape-invariant potential system is assumed to have the form

$$\hat{H}_{\mathrm{P}}^{(q)} = \hbar\Omega\big(\hat{\mathcal{H}}_{+}^{(q)}\hat{\sigma}_{+}\hat{\sigma}_{-} + \hat{\mathcal{H}}_{-}^{(q)}\hat{\sigma}_{-}\hat{\sigma}_{+}\big). \tag{23}$$

We write the atom-potential interaction Hamiltonian as $\hat{H}_{\xi}^{(q)} = \hat{H}_{\Delta} + \hat{\mathcal{W}}_{\xi}^{(q)}$. The term $\hat{\mathcal{W}}_{\xi}^{(q)}$ is constructed using the requirements imposed by the dipole and rotating wave approximations, the form of which reads

$$\hat{\mathcal{W}}_{\xi}^{(q)} = \hbar g\big(\hat{S}_{-}^{(q)}\hat{\sigma}_{+} + \hat{S}_{+}^{(q)}\hat{\sigma}_{-}\big) \tag{24}$$

where g is a real constant coupling strength. The detuning term is given by $\hat{H}_{\Delta} = \hbar\Delta\hat{\sigma}_3$, with Δ being a constant.

3.2. Model superalgebra

Introducing the quantum deformed supercharge operator $\hat{\mathcal{Q}}_q = \hat{S}_-^{(q)}\hat{\sigma}_+$ and its adjoint operator $\hat{\mathcal{Q}}_q^\dagger = \hat{S}_+^{(q)}\hat{\sigma}_-$, it is possible to rewrite the terms $\hat{H}_{\mathrm{P}}^{(q)}$ and $\hat{H}_\xi^{(q)}$ of the coupled system Hamiltonian $\hat{H}_{\mathrm{T}}^{(q)}$ as

$$\hat{H}_{\mathrm{P}}^{(q)} = \hbar\Omega\left\{\hat{\mathcal{Q}}_q, \hat{\mathcal{Q}}_q^\dagger\right\} \qquad \text{and} \qquad \hat{H}_\xi^{(q)} = \hat{H}_\Delta + \hbar g\left(\hat{\mathcal{Q}}_q + \hat{\mathcal{Q}}_q^\dagger\right) \tag{25}$$

and to verify the commutation and anti-commutation relations

$$\left[\hat{\mathcal{Q}}_q, \hat{H}_{\mathrm{P}}^{(q)}\right] = \left[\hat{\mathcal{Q}}_q^\dagger, \hat{H}_{\mathrm{P}}^{(q)}\right] = 0 \qquad \text{and} \qquad \left\{\hat{\mathcal{Q}}_q, \hat{\mathcal{Q}}_q\right\} = \left\{\hat{\mathcal{Q}}_q^\dagger, \hat{\mathcal{Q}}_q^\dagger\right\} = 0. \tag{26}$$

The commutation relations characterize the supersymmetric nature of the quantum deformed Hamiltonian $\hat{H}_{\mathrm{P}}^{(q)}$, with the operators $\hat{\mathcal{Q}}_q$ and $\hat{\mathcal{Q}}_q^\dagger$ as its generators. The anti-commutation relations express the fermionic character of the quantum deformed supercharge operators. Equation (25) closes the graded Lie algebra with the anti-commutators of $\hat{\mathcal{Q}}_q$ with $\hat{\mathcal{Q}}_q^\dagger$. The quantum deformed supercharge $\hat{\mathcal{Q}}_q$ is interpreted as the operator which changes quantum deformed bosonic degrees of freedom into fermionic ones and vice versa. Because of the form presented by the interaction Hamiltonian $\hat{H}_\xi^{(q)}$ in (25), we conclude that the operator $\hat{\mathcal{Q}}_q$ and its adjoint operator $\hat{\mathcal{Q}}_q^\dagger$ are responsible, respectively, for the *heating* and *cooling* process of the coupled system. On the other hand, taking into account commutation relations (26), we can prove that $\left[\hat{H}_\xi^{(q)}, \hat{H}_{\mathrm{P}}^{(q)}\right] = 0$, and thus it is possible to find a common set of eigenstates for these Hamiltonians.

An important extension of the models based on the dipole and rotating wave approximations employs an intensity-dependent interaction between the subsystems [30]. This intensity-dependent interaction makes the enhancement of certain quantum effects possible which would otherwise be difficult to note within the realm of usual interaction model [31, 32]. The intensity-dependent generalization of our model can be constructed by replacing the supercharge and its adjoint operators in the interaction Hamiltonian $\hat{H}_\xi^{(q)}$ by $\hat{\mathcal{Q}}_q \to \hat{S}_-^{(q)}\sqrt{\hat{\mathcal{H}}_-^{(q)}}\hat{\sigma}_+$ and $\hat{\mathcal{Q}}_q^\dagger \to \sqrt{\hat{\mathcal{H}}_-^{(q)}}\hat{S}_+^{(q)}\hat{\sigma}_-$. It is easy to verify that the common eigenstates commutation relation $\left[\hat{H}_\xi^{(q)}, \hat{H}_{\mathrm{P}}^{(q)}\right] = 0$ is preserved in this case.

3.3. Time-evolution operator and the state of the coupled system

By using the Hamiltonian $\hat{H}_{\mathrm{T}}^{(q)}$ presented above we can write the Schrödinger equation for the coupled system as

$$\hat{H}_{\mathrm{T}}^{(q)}|\Psi_q(t)\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi_q(t)\rangle. \tag{27}$$

However, individual terms of the coupled-system total Hamiltonian $\hat{H}_{\mathrm{T}}^{(q)}$ satisfy the general commutation relations $\left[\hat{H}_{\mathrm{A}}, \hat{H}_{\mathrm{P}}^{(q)}\right] = \left[\hat{H}_{\mathrm{P}}^{(q)}, \hat{H}_\Delta\right] = \left[\hat{H}_{\mathrm{P}}^{(q)}, \hat{\mathcal{W}}_\xi^{(q)}\right] = 0$ and, for the case when $\Delta = \frac{1}{2}(\omega_- - \omega_+)$, it is also easy to verify that $\left[(\hat{H}_{\mathrm{A}} + \hat{H}_\Delta), \hat{\mathcal{W}}_\xi^{(q)}\right] = 0$. Therefore, under this resonant condition, if we write the wave state $|\Psi_q(t)\rangle$ as

$$|\Psi_q(t)\rangle = \exp\left(-\mathrm{i}\hat{H}_0^{(q)}t/\hbar\right)|\psi_q(t)\rangle, \qquad \text{with} \qquad \hat{H}_0^{(q)} = \hat{H}_{\mathrm{A}} + \hat{H}_\Delta + \hat{H}_{\mathrm{P}}^{(q)} \tag{28}$$

and insert it into (27) we obtain

$$\hat{\mathcal{W}}_\xi^{(q)}|\psi_q(t)\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi_q(t)\rangle. \tag{29}$$

Introducing the time-evolution operator $\hat{U}_q(t,0)$ such as $|\psi_q(t)\rangle = \hat{U}_q(t,0)|\psi(0)\rangle$, where $|\psi(0)\rangle$ is an arbitrary initial state, and inserting this definition into equation (29), we find

$$\hat{\mathcal{W}}_\xi^{(q)}\hat{U}_q(t,0) = \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{U}_q(t,0). \tag{30}$$

Since the Hamiltonian term $\hat{\mathcal{W}}_\xi^{(q)}$ is time independent, the solution of (30) satisfying the initial condition $\hat{U}_q(0,0) = \hat{\mathbb{1}}$ can be written formally as $\hat{U}_q(t,0) = \exp(-\mathrm{i}\hat{\mathcal{W}}_\xi^{(q)}t/\hbar)$. Using the series expansion of the exponential function, expression (24) for $\hat{\mathcal{W}}_\xi^{(q)}$ and the properties of the $\hat{\sigma}_\pm$-operators, it is possible to obtain the analytic expression for the time-evolution operator

$$\begin{aligned}\hat{U}_q(t,0) = {}& \hat{\sigma}_+\hat{\sigma}_-\cos\left(g\hat{\mu}_+^{(q)}t\right) + \hat{\sigma}_-\hat{\sigma}_+\cos\left(g\hat{\mu}_-^{(q)}t\right)\\ &+\hat{\sigma}_+\hat{\mathcal{K}}_q^\dagger\{\sin(g\hat{\mu}_-^{(q)}t)\} - \hat{\sigma}_-\{\sin(g\hat{\mu}_-^{(q)}t)\}\hat{\mathcal{K}}_q\end{aligned} \tag{31}$$

where

$$\hat{\mu}_\pm^{(q)} = \sqrt{\hat{\mathcal{H}}_\pm^{(q)}} \qquad \text{and} \qquad \hat{\mathcal{K}}_q = \frac{\mathrm{i}}{\sqrt{\hat{\mathcal{H}}_-^{(q)}}}\hat{S}_+^{(q)}. \tag{32}$$

Note that in the intensity-dependent model the operators $\hat{\mu}_\pm^{(q)}$ in the argument of the trigonometric functions in (31) must be replaced by $\hat{\mu}_\pm^{(q)} \to \hat{\mathcal{H}}_\pm^{(q)}$. Using the expression of the time-evolution operator $\hat{U}_q(t,0)$ we can write the final expression for the wave state of the coupled system as

$$|\Psi_q(t)\rangle = \exp\left(-\mathrm{i}\hat{H}_0^{(q)}t/\hbar\right)\hat{U}_q(t,0)|\psi(0)\rangle, \tag{33}$$

which is still valid for any quantum-deformed shape-invariant potential system.

3.4. The density operator

A simple and elegant way of incorporating statistical distributions of the initial conditions into quantum dynamics of the coupled system is to represent the state of the quantum system by using the Hermitian density operator, defined as $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. At any time $t > 0$, the time evolution of $\hat{\rho}(t)$ is given by the Liouville equation of motion $\mathrm{i}\hbar\mathrm{d}\hat{\rho}(t)/\mathrm{d}t = [\hat{H}(t),\hat{\rho}(t)]$. Knowledge of $\hat{\rho}(t)$ enables us to obtain the expectation value of any observable $\hat{O}$ through

$$\langle\hat{O}(t)\rangle = \frac{\mathrm{Tr}\{\hat{\rho}(t)\hat{O}\}}{\mathrm{Tr}\{\hat{\rho}(t)\}}. \tag{34}$$

Applying this formalism to our quantum deformed coupled problem we use the state vector (33) and the commutation relations presented at the beginning of the previous section, involving the terms of total Hamiltonian $\hat{H}_0^{(q)}$, to obtain

$$\begin{aligned}\hat{\rho}(q;t) = {}& \exp\left(-\mathrm{i}\hat{H}_\mathrm{P}^{(q)}t/\hbar\right)\hat{U}_q(t,0)\hat{\rho}_0\hat{U}_q^\dagger(t,0)\exp\left(\mathrm{i}\hat{H}_\mathrm{P}^{(q)}t/\hbar\right), \qquad \text{where}\\ &\hat{\rho}_0 = |\psi(0)\rangle\langle\psi(0)|.\end{aligned} \tag{35}$$

In the analysis of the dynamics of the coupled system it is very instructive to assume that at time $t = 0$ its quantum state is uncorrelated, i.e., it is described by a pure state obtained as a direct product $|\psi(0)\rangle = |\beta\rangle \otimes |\varphi\rangle$ of the initial states $|\beta\rangle$ of two-level atom and $|\varphi\rangle$ the quantum deformed coupling potential. Many details of the dynamics of the coupled system strongly depend on its initial condition, and in order to understand the global influence of the quantum-deformed shape-invariant potential on the system dynamics we consider at $t = 0$ that the two-level atom is in the lower state $|\beta\rangle = |-\rangle$ and the quantum deformed shape-invariant potential system is in a purely coherent state $|\varphi\rangle = |z\rangle_\mathrm{C}$ or, as a second possibility, in a purely squeezed state $|\varphi\rangle = |z\rangle_{\rm S}$. (It is worth recalling that coherent states are considered to be the quantum states which most closely approach the classical limit.) The purely coherent [33] and squeezed [34] states for quantum deformed shape-invariant systems can be obtained, in a generalized way, by an expansion in the basis $\{|n\rangle; n = 0, 1, 2, \ldots\}$

$$|z;q;a_r\rangle_{\rm C} = \sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n^{(\rm C)}(q;a_r)}\right\}|n\rangle \qquad \text{and} \qquad |z;q;a_r\rangle_{\rm S} = \sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n^{(\rm S)}(q;a_r)}\right\}|2n\rangle \tag{36}$$

where the expansion coefficients $h_n^{(\rm C,S)}(q;a_r)$ are given by $h_0^{(\rm C,S)}(q;a_r) = 1$, and for $n \geqslant 1$,

$$h_n^{(\rm C)}(q;a_r) = \prod_{k=0}^{n-1}\left\{\frac{\sqrt{\Theta_{nk}^{(\rm C)}(q)}}{\mathcal{Z}_{r+k}^{(q)}}\right\}, \qquad \text{with} \qquad \Theta_{nk}^{(\rm C)}(q) = q^{2R(a_0)}\mathcal{F}_q^2 q^{e_n+e_k}[e_n - e_k]_q \tag{37}$$

while

$$h_n^{(\rm S)}(q;a_r) = \prod_{k=0}^{n-1}\left\{\frac{\sqrt{\Theta_{nk}^{(\rm S)}(q)}}{\mathcal{Z}_{r+2k}^{(q)}}\right\}, \qquad \text{with} \qquad \Theta_{nk}^{(\rm S)}(q) = q^{-R(a_{2k+1})}\frac{[e_{2n} - e_{2k}]_q}{[e_{2n} - e_{2k+1}]_q}. \tag{38}$$

In these expressions $\mathcal{Z}_{r+k}^{(q)} = \hat{T}^k\mathcal{Z}_r^{(q)}\hat{T}^{\dagger k}$, where $\mathcal{Z}_r^{(q)} \equiv \mathcal{Z}(q;a_1,a_2,a_3,\ldots)$ is an arbitrary complex functional. Under these assumptions the initial state of the coupled system is described by

$$|\psi(0)\rangle_{\rm C} = \sum_{n=0}^{\infty} b_n^{(\rm C)}(q;a_r)|n\rangle \otimes |-\rangle \qquad \text{or} \qquad |\psi(0)\rangle_{\rm S} = \sum_{n=0}^{\infty} b_n^{(\rm S)}(q;a_r)|2n\rangle \otimes |-\rangle, \tag{39}$$

where

$$b_n^{(\rm X)}(q;a_r) = \frac{z^n}{h_n^{(\rm X)}(q;a_r)} \in \mathbb{C}, \qquad \text{with} \qquad \rm X = C \quad \text{or S}. \tag{40}$$

Using the property

$$\hat{S}_{\pm}^{(q)} f\left(\hat{\mathcal{H}}_{\pm}^{(q)}\right) = f\left(\hat{\mathcal{H}}_{\mp}^{(q)}\right)S_{\pm}^{(q)}, \tag{41}$$

valid for any analytical function $f(x)$, the resonant condition $\Delta = \frac{1}{2}(\omega_- - \omega_+)$, and expression (39) to get $\hat{\rho}_0^{(\rm X)}$, it is possible to show that the time-evolved density operator (35) can be explicitly expressed in the matrix form

$$\hat{\boldsymbol{\rho}}^{(\rm X)}(q;t) = \frac{1}{\mathcal{N}_{\rm X}^{(q)}}\begin{bmatrix} \left|\mathcal{D}_{\rm X}^{(q)}(t)\right\rangle\left\langle\mathcal{D}_{\rm X}^{(q)}(t)\right| & \left|\mathcal{D}_{\rm X}^{(q)}(t)\right\rangle\left\langle\mathcal{C}_{\rm X}^{(q)}(t)\right| \\ \left|\mathcal{C}_{\rm X}^{(q)}(t)\right\rangle\left\langle\mathcal{D}_{\rm X}^{(q)}(t)\right| & \left|\mathcal{C}_{\rm X}^{(q)}(t)\right\rangle\left\langle\mathcal{C}_{\rm X}^{(q)}(t)\right| \end{bmatrix} \tag{42}$$

where the factor

$$\mathcal{N}_{\rm X}^{(q)} = {}_{\rm X}\langle\psi(0)|\psi(0)\rangle_{\rm X} = \sum_{n=0}^{\infty}\left|b_n^{(\rm X)}(q;a_r)\right|^2 \tag{43}$$

was introduced to satisfy the normalization condition $\mathrm{Tr}\{\hat{\rho}^{(\rm X)}(q;t)\} = 1$. The time-dependent states which compose the elements of the matrix $\hat{\boldsymbol{\rho}}^{(\rm X)}(q;t)$ are expressed by

$$\begin{aligned} \left|\mathcal{C}_{\rm X}^{(q)}(t)\right\rangle &= \mathrm{e}^{-\mathrm{i}\Omega\hat{\mathcal{H}}_-^{(q)}t}\cos\left(g\hat{\mu}_-^{(q)}t\right)|z;q;a_r\rangle_{\rm X} \qquad \text{and} \\ \left|\mathcal{D}_{\rm X}^{(q)}(t)\right\rangle &= \hat{\mathcal{K}}_q^\dagger\,\mathrm{e}^{-\mathrm{i}\Omega\hat{\mathcal{H}}_-^{(q)}t}\sin\left(g\hat{\mu}_-^{(q)}t\right)|z;q;a_r\rangle_{\rm X}. \end{aligned} \tag{44}$$

In order to explore the dynamics of each subsystem which composes the coupled system we need to calculate, from the density operator $\hat{\rho}^{(\mathrm{X})}(q;t)$, the reduced density operator for either the atom or the quantum deformed coupling potential. Tracing out the coupling potential degrees of freedom $\hat{\rho}_{\mathrm{A}}^{(\mathrm{X})}(q;t) = \mathrm{Tr}_{\mathrm{P}}\{\hat{\rho}^{(\mathrm{X})}(q;t)\}$, we obtain the reduced 2×2 atomic density matrix $\hat{\boldsymbol{\rho}}_{\mathrm{A}}^{(\mathrm{X})}(q;t)$ whose elements are given by

$$\left\{\hat{\boldsymbol{\rho}}_{\mathrm{A}}^{(\mathrm{X})}(q;t)\right\}_{jk} = \sum_{n=0}^{\infty}\langle n|\{\hat{\boldsymbol{\rho}}^{(\mathrm{X})}(q;t)\}_{jk}|n\rangle. \tag{45}$$

Similarly the coupling potential reduced density operator, obtained from $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t) = \mathrm{Tr}_{\mathrm{A}}\{\hat{\rho}^{(\mathrm{X})}(q;t)\}$, gives

$$\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t) = \{\hat{\boldsymbol{\rho}}^{(\mathrm{X})}(q;t)\}_{11} + \{\hat{\boldsymbol{\rho}}^{(\mathrm{X})}(q;t)\}_{22} = \frac{1}{\mathcal{N}_{\mathrm{X}}^{(q)}}\left\{\left|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right\rangle\left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right| + \left|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle\left\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right|\right\}. \tag{46}$$

4. Temporal behavior of the quantum dynamical variables

4.1. Population inversion factor

The simplest nontrivial physical quantity used to analyse the quantum dynamic behavior of a coupled two-level system is the population inversion factor. This quantity, also called the degree of excitation of the system, is defined as $\hat{\mathrm{W}} \equiv \hat{\sigma}_+\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+ = \hat{\sigma}_3$, and represents the difference between the population of the excited and the ground atomic states. In this case, inserting the time-evolved density operator (35) into equation (34) and taking into account the commutation relation between $\hat{H}_{\mathrm{P}}^{(q)}$ and $\hat{\sigma}_3$, we obtain the expectation value

$$\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle = \frac{{}_{\mathrm{X}}\langle\psi(0)|\hat{\rho}_0^{(\mathrm{X})}\hat{U}_q^{\dagger}(t,0)\hat{\sigma}_3\hat{U}_q(t,0)|\psi(0)\rangle_{\mathrm{X}}}{{}_{\mathrm{X}}\langle\psi(0)|\hat{\rho}_0^{(\mathrm{X})}|\psi(0)\rangle_{\mathrm{X}}}. \tag{47}$$

By using property (41) and equation (31) for $\hat{U}_q(t,0)$ in (47), we can show that

$$\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle = {}_{\mathrm{X}}\left\langle\psi(0)\left|\begin{bmatrix}\cos(2g\hat{\mu}_+^{(q)}t) & \hat{\mathcal{K}}_q^{\dagger}\{\sin(2g\hat{\mu}_-^{(q)}t)\}\\ \{\sin(2g\hat{\mu}_-^{(q)}t)\}\hat{\mathcal{K}}_q & -\cos(2g\hat{\mu}_-^{(q)}t)\end{bmatrix}\right|\psi(0)\right\rangle_{\mathrm{X}}\Big/{}_{\mathrm{X}}\langle\psi(0)|\psi(0)\rangle_{\mathrm{X}} \tag{48}$$

and when we consider the initial state of the system (39) in (48) we find the general expression

$$\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle = -\sum_{n,n'=0}^{\infty} b_n^{(\mathrm{X})*}(q;a_r)b_{n'}^{(\mathrm{X})}(q;a_r)\langle n|\cos(2g\hat{\mu}_-^{(q)}t)|n'\rangle\Big/\sum_{n=0}^{\infty}\left|b_n^{(\mathrm{X})}(q;a_r)\right|^2. \tag{49}$$

To obtain a final expression we use the series expansion of the cosine function, expressions (12), (32) and the commutation between any function of a_n and the operators $\hat{\mathcal{H}}_{\pm}^{(q)}$, to get

$$\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle = -\sum_{n=0}^{\infty} p_n^{(\mathrm{X})}(q)\cos\left\{2\theta_n^{(\mathrm{X})}(q;t)\right\} \qquad \text{where}$$
$$p_n^{(\mathrm{X})}(q) = \left|b_n^{(\mathrm{X})}(q;a_r)\right|^2\Big/\sum_{n=0}^{\infty}\left|b_n^{(\mathrm{X})}(q;a_r)\right|^2. \tag{50}$$

In this case, the cosine function argument factors $\theta_n^{(\mathrm{X})}(q;t)$ are given by

$$\theta_n^{(\mathrm{C})}(q;t) = gt\sqrt{\mathcal{E}_n^{(q)}} \qquad \text{and} \qquad \theta_n^{(\mathrm{S})}(q;t) = gt\sqrt{\mathcal{E}_{2n}^{(q)}}. \tag{51}$$

To conclude this section we observe that:

- the dynamics of $\langle \hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle$, given by relation (50), is obtained as a sum of contributions from which the time-independent weight $p_n^{(\mathrm{X})}(q)$ is determined by the initial state of the coupling potential. The influence of the interaction appears in these contributions only in the argument factors $\theta_n^{(\mathrm{X})}(q;t)$ of the time-dependent circular functions;
- the weight $p_n^{(\mathrm{X})}(q)$ gives the probability of finding the eigenstate $|n\rangle$ of the Hamiltonian $\hat{\mathcal{H}}_-^{(q)}$ in the quantum deformed coherent or squeezed states $|z;q;a_r\rangle_{\mathrm{C}}$ and $|z;q;a_r\rangle_{\mathrm{S}}$ and corresponds to the simplest statistical property that can be evaluated about these states;
- the results for the intensity-dependent interaction model are obtained by just replacing the cosine function argument factors by the expressions $\theta_n^{(\mathrm{C})}(q;t) \rightarrow gt\mathcal{E}_n^{(q)}$ and $\theta_n^{(\mathrm{S})}(q;t) \rightarrow gt\mathcal{E}_{2n}^{(q)}$.

4.2. Entropy and entanglement of the system

The entropy in quantum mechanics [35] is defined in terms of the density operator as $S = -\mathrm{Tr}\{\hat{\rho}\ln\hat{\rho}\}$. If $\hat{\rho}$ describes a pure state this entropy vanishes ($S = 0$), while if $\hat{\rho}$ describes a mixed state then $S \neq 0$. Therefore, entropy offers a quantitative measure of the disorder of a system, and of the purity of a quantum state. The higher the entropy the greater the entanglement of the system. Since for a closed system the total entropy S is constant, in the case of a composite system we study the partial entropies of system components, such as the atom and the quantum-deformed coupling potential subsystems. These partial entropies are defined through the corresponding reduced density operators by

$$\begin{aligned} S_{\mathrm{A}}^{(\mathrm{X})}(q;t) &= -\mathrm{Tr}_{\mathrm{A}}\big\{\hat{\rho}_{\mathrm{A}}^{(\mathrm{X})}(q;t)\ln\big[\hat{\rho}_{\mathrm{A}}^{(\mathrm{X})}(q;t)\big]\big\} \qquad \text{and} \\ S_{\mathrm{P}}^{(\mathrm{X})}(q;t) &= -\mathrm{Tr}_{\mathrm{P}}\big\{\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)\ln\big[\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)\big]\big\}. \end{aligned} \tag{52}$$

Note that the operation of tracing over part of the variables of the whole system means that $\hat{\rho}_{\mathrm{A(P)}}^{(\mathrm{X})}(q;t)$ is no longer governed by a unitary time evolution and consequently $S_{\mathrm{A(P)}}^{(\mathrm{X})}(q;t)$ is no longer time independent. This implies that the composite system can evolve from a pure to a mixed state and vice versa with oscillations in the subsystem entropy.

Taken as a whole, the two-level atom coupled to a one-dimensional quantum-deformed shape-invariant system in an overall pure state constitutes a bipartite quantum system in a Hilbert space with the tensor product structure $\mathrm{E} = \mathrm{E}_{\mathrm{A}} \otimes \mathrm{E}_{\mathrm{P}}$. For these conditions the Araki and Lieb theorem [36] is valid and if the combined system begins as a pure quantum state (that is, the total entropy of the system is equal to zero), then at $t > 0$ the partial entropies of the subsystems are precisely equal. On the other hand, since the trace of an operator depends only on its eigenvalues and is invariant under a similarity transformation, we can go to a basis in which $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)$ is diagonal to evaluate the partial entropy of the coupling potentials using

$$S_{\mathrm{P}}^{(\mathrm{X})}(q;t) = -\big\{\lambda_-^{(\mathrm{X})}(q;t)\ln\big[\lambda_-^{(\mathrm{X})}(q;t)\big] + \lambda_+^{(\mathrm{X})}(q;t)\ln\big[\lambda_+^{(\mathrm{X})}(q;t)\big]\big\} \tag{53}$$

obtained from equation (52), where $\lambda_\pm^{(\mathrm{X})}(q;t)$ are the eigenvalues of $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)$. Considering the eigenvalue equation $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)\big|\zeta_\pm^{(\mathrm{X})}(q;t)\big\rangle = \lambda_\pm^{(\mathrm{X})}(q;t)\big|\zeta_\pm^{(\mathrm{X})}(q;t)\big\rangle$ and expression (46) of $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)$ we expect that eigenstates have the form $\big|\zeta_\pm^{(\mathrm{X})}(q;t)\big\rangle = \alpha_{\mathcal{C}}\big|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\big\rangle + \alpha_{\mathcal{D}}\big|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\big\rangle$ so that

$$\begin{aligned} \hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)\big|\zeta_\pm^{(\mathrm{X})}(q;t)\big\rangle = {}& \frac{1}{\mathcal{N}_{\mathrm{X}}^{(q)}}\left(\big\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\big|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\big\rangle + \frac{\alpha_{\mathcal{D}}}{\alpha_{\mathcal{C}}}\big\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\big|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\big\rangle\right)\alpha_{\mathcal{C}}\big|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\big\rangle \\ & + \frac{1}{\mathcal{N}_{\mathrm{X}}^{(q)}}\left(\big\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\big|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\big\rangle + \frac{\alpha_{\mathcal{C}}}{\alpha_{\mathcal{D}}}\big\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\big|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\big\rangle\right)\alpha_{\mathcal{D}}\big|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\big\rangle \end{aligned} \tag{54}$$

and, consequently for $\left|\zeta_{\pm}^{(\mathrm{X})}(q;t)\right\rangle$ to be an eigenstate of $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X})}(q;t)$, we need to satisfy the condition

$$\mathcal{N}_{\mathrm{X}}^{(q)}\lambda_{\pm}^{(\mathrm{X})}(q;t) = \left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right\rangle + \frac{\alpha_{\mathcal{D}}}{\alpha_{\mathcal{C}}}\left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle$$
$$= \left\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle + \frac{\alpha_{\mathcal{C}}}{\alpha_{\mathcal{D}}}\left\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right\rangle. \quad (55)$$

Using this relation and that $\left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right\rangle + \left\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle = \mathcal{N}_{\mathrm{X}}^{(q)}$ we can show, after some calculations, that

$$\lambda_{\pm}^{(\mathrm{X})}(q;t) = \frac{1}{2}\left(1 \pm \frac{1}{\mathcal{N}_{\mathrm{X}}^{(q)}}\sqrt{\left\{\left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right\rangle - \left\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle\right\}^2 + 4\left|\left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle\right|^2}\right). \quad (56)$$

To get the final expression for the factors that appear in (56), we can use the relation $\hat{\mathcal{K}}_q\hat{\mathcal{K}}_q^{\dagger} = \hat{\mathcal{K}}_q^{\dagger}\hat{\mathcal{K}}_q = \hat{\mathbb{1}}$ and equations (44) and (50) to obtain

$$\frac{1}{\mathcal{N}_{\mathrm{X}}^{(q)}}\left\{\left\langle\mathcal{C}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{C}_{\mathrm{X}}^{(q)}(t)\right\rangle - \left\langle\mathcal{D}_{\mathrm{X}}^{(q)}(t)\middle|\mathcal{D}_{\mathrm{X}}^{(q)}(t)\right\rangle\right\} = -\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle. \quad (57)$$

With the help of equations (12), (17), (32), (36), (37) and (44), we can also show that $\langle\mathcal{C}_{\mathrm{S}}^{(q)}(t)|\mathcal{D}_{\mathrm{S}}^{(q)}(t)\rangle = 0$ and

$$\frac{1}{\mathcal{N}_{\mathrm{C}}^{(q)}}\left|\left\langle\mathcal{C}_{\mathrm{C}}^{(q)}(t)\middle|\mathcal{D}_{\mathrm{C}}^{(q)}(t)\right\rangle\right| = \left|z\mathcal{Z}_{r-1}^{(q)}\sum_{n=0}^{\infty}\frac{p_n^{(\mathrm{C})}(q)}{\phi_n^{(q,\mathrm{C})}}\cos\left\{\theta_n^{(\mathrm{C})}(q;t)\right\}\sin\left\{\phi_n^{(\mathrm{C})}(q;t)\right\}\right| \qquad \text{where}$$
$$\phi_n^{(\mathrm{C})}(q;t) = gt\sqrt{\mathcal{A}_n^{(q)}} \quad (58)$$

where we took into account the auxiliary relations $\mathcal{A}_n^{(q)} - \mathcal{E}_n^{(q)} = \mathcal{A}_0^{(q)}$ and $\hat{T}^{\dagger}\mathcal{E}_{n+1}^{(q)}\hat{T} = \mathcal{A}_n^{(q)}$, as well as $\hat{T}^{\dagger}b_{n+1}^{(\mathrm{C})}(q;a_r)\hat{T} = z\mathcal{Z}_{r-1}^{(q)}b_n^{(\mathrm{C})}(q;a_r)\big/\sqrt{\mathcal{A}_n^{(q)}}$, which can be established after some algebra.

The partial entropy for the case of an intensity-dependent interaction Hamiltonian can be obtained with the same procedure presented in the previous section by replacing the factors: $\theta_n^{(\mathrm{C})}(q;t) \to gt\mathcal{E}_n^{(q)}$ and $\theta_n^{(\mathrm{S})}(q;t) \to gt\mathcal{E}_{2n}^{(q)}$.

We note that to apply our general approach for a given quantum deformed and shape-invariant coupling potential we need to specify only the eigenvalue spectra $\mathcal{A}_n^{(q)}$ and $\mathcal{E}_n^{(q)}$ as well as the expansion coefficients $b_n^{(\mathrm{X})}(q;a_r)$ related to the initial state of the system. Using these informations we obtain the function argument factors $\theta_n^{(\mathrm{X})}(q;t)$ and $\phi_n^{(\mathrm{C})}(q;t)$ and the expansion weight $p_n^{(\mathrm{X})}(q)$ to be used in the calculation of the observables $\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle$ and $S_{\mathrm{P}}^{(\mathrm{X})}(q;t)$. Due to the higher dimensionality of the problem it is not possible to obtain simple analytical expressions for the series related to the observables $\langle\hat{\mathrm{W}}_{\mathrm{X}}(q;t)\rangle$ and $S_{\mathrm{P}}^{(\mathrm{X})}(q;t)$, requiring a numerical approach. The numerical evaluation of series is a nontrivial problem and in the application of the next section we used the Smith [37] routines package of multiple precise computation.

5. An application for a quantum deformed Pösch–Teller potential system

As a specific example we consider in this section the coupling of the two-level atom with a quantum deformed Pösch–Teller Potential system. This potential was originally introduced in a molecular physics context [38] and is closely related to several other potentials widely used in molecular and solid state physics. Besides that the Pöschl–Teller potential, in its trigonometric form, presents the interesting property of represents the infinite square well as a special limit [39]. We consider the partner potentials $V_{\pm}(x)$ for the trigonometric Pöschl–Teller system [40] obtained with the superpotential $W(x, a_1) = \sqrt{\hbar\Omega}(\beta a_1 \cot\beta x + \delta \csc\beta x)$ where a_1, β and δ are real constants. In this case the remainders involved with the shape invariance condition (3) are given by $R(a_1) = \beta^2\eta(2a_1 + \eta)$ and the potential parameters are related by $a_{n+1} = a_n + \eta$, with $\eta = \sqrt{\hbar/(2M\Omega)}$. Using these facts in (4) we find the eigenvalue factor e_n with the form $e_n = \kappa n(n + \gamma)$ where $\kappa = (\beta\eta)^2$ and $\gamma = 2a_1/\eta$. Considering the above remainder relation, it is easy of verify that if we define $\mathcal{F}_q = q^{\beta^2 a_0^2}$, then the constraint presented in (8), related to the preservation of the shape invariance of the primary system after the quantum deformation process, is satisfied because

$$q^{R(a_0)}\mathcal{F}_q = q^{\beta^2(a_1^2-a_0^2)}q^{\beta^2 a_0^2} = q^{\beta^2 a_1^2} = \hat{T}q^{\beta^2 a_0^2}\hat{T}^\dagger = \hat{T}\mathcal{F}_q\hat{T}^\dagger. \tag{59}$$

Then using the expression of e_n we can calculate the eigenvalues $\mathcal{A}_n^{(q)}$ and $\mathcal{E}_n^{(q)}$ in (14) and (15), obtaining

$$\begin{aligned} \mathcal{A}_n^{(q)} &= q^{\frac{1}{2}\kappa(\gamma-2)^2} q^{\kappa(n+1)(n+\gamma-1)}[\kappa(n+1)(n+\gamma-1)]_q \qquad \text{and} \\ \mathcal{E}_n^{(q)} &= q^{\frac{1}{2}\kappa\gamma^2} q^{\kappa n(n+\gamma)}[\kappa n(n+\gamma)]_q. \end{aligned} \tag{60}$$

Similarly using the expressions of e_n and $\mathcal{F}_q$ in (37) and (38) we can show that

$$\prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(\mathrm{S})}(q)} = q^{-\kappa n(2n+\gamma+1)}\sqrt{\frac{(\gamma, 0; q^{8\kappa})_n}{(\gamma, \frac{1}{2}; q^{8\kappa})_n}} \qquad \text{and} \tag{61}$$

$$\prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(\mathrm{C})}(q)} = q^{\frac{1}{2}n\{\kappa[\frac{1}{3}(n-1)(2n+3\gamma-1)+\frac{1}{2}\gamma^2]+1\}}\sqrt{\frac{(\gamma, 0; q^{2\kappa}x)_n}{(1-q^2)^n}} \tag{62}$$

where we used the two-parameter generalization of the q-shifted factorial $(a, b; q)_n$, defined as [33, 34]

$$(a, b; q)_0 = 1 \qquad \text{and} \qquad (a, b; q)_n = \prod_{k=0}^{n-1}\left(1 - q^{[n(n+a)-(k+b)(k+a+b)]}\right), \qquad \text{when} \quad n = 1, 2, 3, \ldots \tag{63}$$

To explore the construction of purely coherent states for quantum-deformed shape-invariant systems, we introduce the generalizing functional with the simple form $\mathcal{Z}_s^{(q)} = \mathcal{Z}_q q^{\alpha R^2(a_1)}$, where α and $\mathcal{Z}_q$ are real constants. Using the recurrence relation $R(a_k) = \kappa(2k + \gamma - 1)$ we find that

$$\begin{aligned} \prod_{k=0}^{n-1}\mathcal{Z}_{r+k}^{(q)} &= \mathcal{Z}_q^n q^{\alpha\kappa^2 n[\frac{2}{3}(n-1)(2n+3\gamma+2)+(\gamma+1)^2]} \qquad \text{and} \\ \prod_{k=0}^{n-1}\mathcal{Z}_{r+2k}^{(q)} &= \mathcal{Z}_q^n q^{4\alpha\kappa^2 n[\frac{1}{3}(n-1)(4n+3\gamma+1)+\frac{1}{4}(\gamma+1)^2]}. \end{aligned} \tag{64}$$

Taking into account these results and using (62) in (37) and (61) in (38) we find for the expansion coefficients (40) the expressions

$$\begin{aligned} \left|b_n^{(\mathrm{C})}(q; a_r)\right|^2 &= \frac{[|z\mathcal{Z}_q|^2(1-q^2)q^{\varrho_c}]^n q^{\vartheta_n^{(c)}}}{(\gamma, 0; q^{2\kappa})_n} \qquad \text{and} \\ \left|b_n^{(\mathrm{S})}(q; a_r)\right|^2 &= \frac{[|z\mathcal{Z}_q|^2 q^{\varrho_s}]^n q^{\vartheta_n^{(s)}}(\gamma, \frac{1}{2}; q^{8\kappa})_n}{(\gamma, 0; q^{8\kappa})_n}, \end{aligned} \tag{65}$$

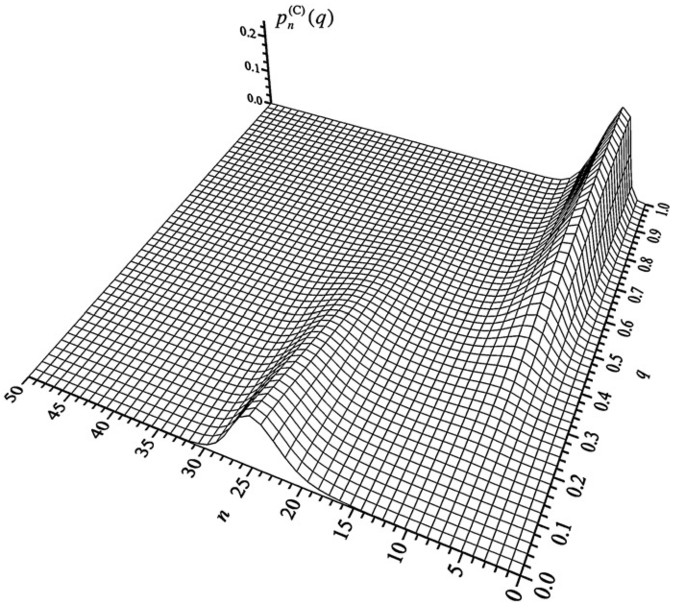

Figure 1. The distribution of probability $p_n^{(C)}(q)$ of finding the deformed coupling potential system in the quantum state $|n\rangle$ as a function of the deformation parameter q.

where

$$\begin{cases} \varrho_{\rm c} = \dfrac{\kappa}{2}\left[(\delta-1)\left(\gamma^2-\dfrac{1}{3}\right)+2\gamma-1\right]-1, & \vartheta_n^{(\rm c)} = \kappa n^2\left[(\delta-1)\left(\dfrac{2}{3}n+\gamma\right)+1\right] \\ \varrho_{\rm s} = \dfrac{\kappa}{2}\left\{\delta\left[\left(\dfrac{\gamma-1}{2}\right)^2-\dfrac{1}{3}\right]+4(\gamma+1)\right\}, & \vartheta_n^{(\rm s)} = \kappa n^2\left\{\dfrac{2}{3}\delta n+\left[\dfrac{1}{2}\delta(\gamma-1)+4\right]\right\} \end{cases} \tag{66}$$

with $\delta = 4\alpha\kappa$.

In the figures we present the numerical results obtained for this application using the set of constants: $\alpha = 26, \gamma = 1.01, \kappa = 0.01$ and $z\mathcal{Z}_q = 1$. Figure 1 displays the distribution of probability $p_n^{(C)}(q)$ of finding the deformed coupling potential system in the quantum state $|n\rangle$ as a function of the deformation parameter q, when the initial state of the potential system is the purely coherent state. It should be noted that $p_n^{(C)}(q)$ shows: (i) a broader and lower distribution centered in $n \approx 26$ for lower values of the deformation parameter $(0.0 < q < 0.2)$; (ii) a sharper and higher distribution centered in $n \approx 5$ for higher values of the deformation parameter $(0.6 < q < 1.0)$; (iii) two lower concentration regions centered in $n \approx 5$ and $n \approx 26$ for intermediate values of the deformation parameter $(0.2 < q < 0.6)$. This change in the behavior of $p_n^{(C)}(q)$ with the values of q will be responsible for interesting behavior of the atomic population inversion factor and the potential partial entropy.

In figure 2 we plot, in a three-dimensional *waterfall* layout, the population inversion factor $\langle\hat{\rm W}_{\rm C}(q;t)\rangle$ in terms of the time variable $\tau = 2gt$ for the deformation parameter values $q = 0.2, 0.4, 0.6$ and 0.999. To make comparison of the results obtained using different values of q easier, we fill the area under each curve. To understand the time behavior of $\langle\hat{\rm W}_{\rm C}(q;t)\rangle$, we observe that each term in the sum (50) has a different frequency, and as the time increases they become uncorrelated and interfere destructively, causing a *collapse* $[\langle\hat{\rm W}_{\rm C}(q;t)\rangle \approx 0]$.

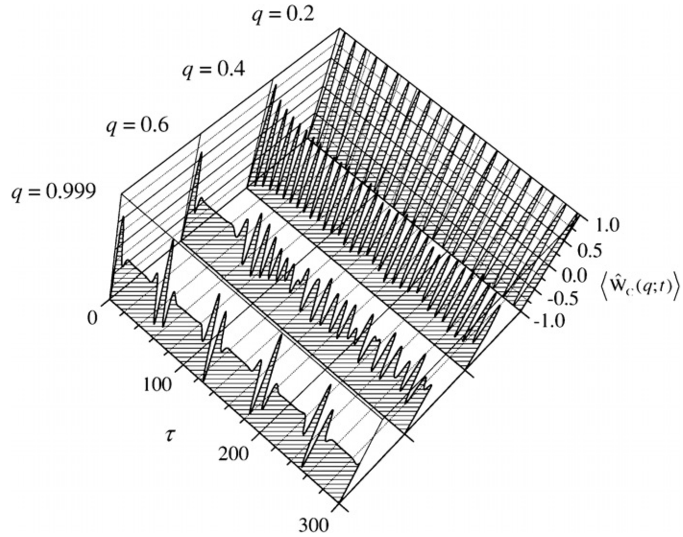

Figure 2. The population inversion factor $\langle \hat{W}_C(q;t)\rangle$ in terms of the time variable $\tau = 2gt$ calculated for the deformation parameter values $q = 0.2, 0.4, 0.6$ and 0.999.

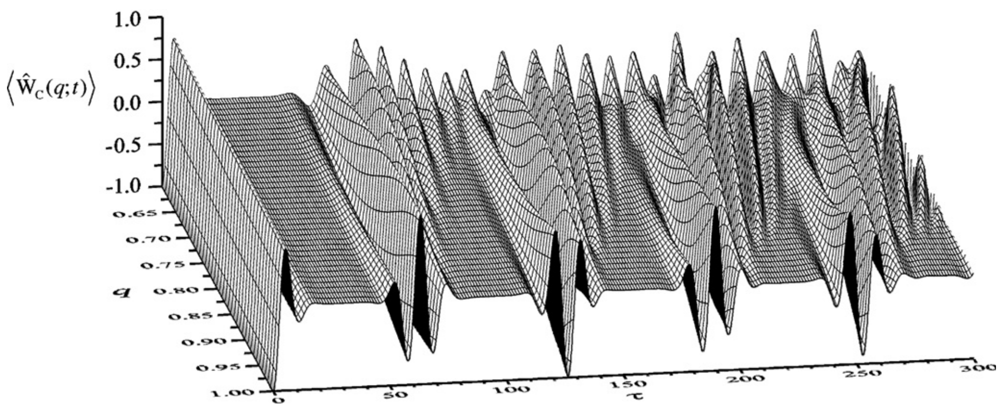

Figure 3. The three-dimensional plot of the population inversion factor $\langle \hat{W}_C(q;t)\rangle$, as a function of the time variable τ and the deformation parameter q.

The discrete character of the sum over the quantum states in the coherent states ensures that, after some finite time, all the oscillating terms come back almost in phase with each other, restoring the coherent oscillations and creating periodic *revivals* (periodic packets of finite $\langle \hat{W}_C(q;t)\rangle$ oscillations). However, as the frequencies are not necessarily integers and thus may be incommensurate, the re-phasing is not perfect and the revivals get broader and broader. The periodic behavior of the cosine function in time, the expression of its argument and its dependence on the coupling potentials e_n factors define the form and the periodicity of these events in $\langle \hat{W}_C(q;t)\rangle$. In our coupled system, appearance of time periodic collapse and revival events in $\langle \hat{W}_C(q;t)\rangle$ is well defined when $q \approx 1$. With the reduction of the value of q the number of events is getting reduced and concentrated in the region of lower values of τ. For higher values of τ the pattern with collapse and revival events is gradually substituted by

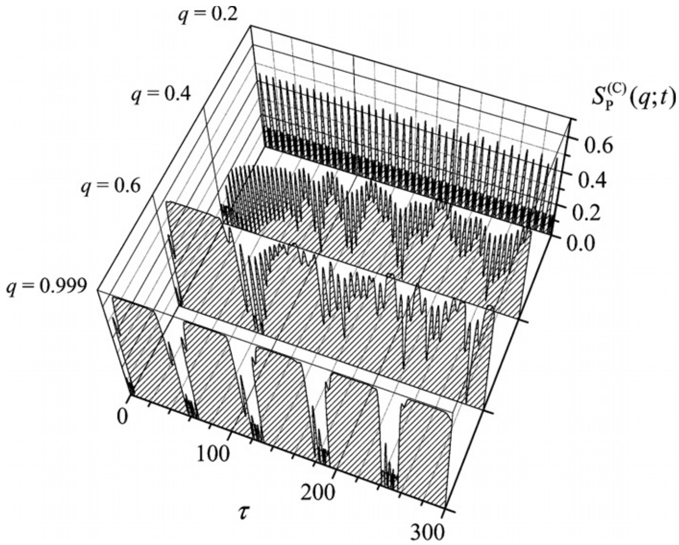

Figure 4. Time evolution of the coupling potential partial entropy $S_P^{(C)}(q;t)$, calculated for the deformation parameter values $q = 0.2$, 0.4, 0.6 and 0.999.

a complex oscillating pattern. This tendency is clearly visualized in figure 3 that shows a three-dimensional plot of the population inversion factor $\langle \hat{W}_C(q;t)\rangle$ as a function of τ and q with $(0.6 \leqslant q \leqslant 0.999)$. For q values lower than 0.5 the collapse and revival events disappear being substituted by Rabi oscillations with a non-defined envelope pattern. Finally, for very low values of the deformation parameter ($q < 0.3$) these resulting Rabi oscillations in $\langle \hat{W}_C(q;t)\rangle$ assume a standard pattern with a constant unity amplitude. The behavior of the distribution of probability $p_n^{(C)}(q)$ with q shown in figure 1 is essential to understand the behavior of $\langle \hat{W}_C(q;t)\rangle$ observed in figures 2 and 3.

In figure 4, we show, also in a three-dimensional waterfall layout, the coupling potential partial entropy $S_P^{(C)}(q;t)$ in terms of the time variable $\tau = 2gt$ for the same set of values of the deformation parameter q used in figure 2 for the $\langle \hat{W}_C(q;t)\rangle$ case. As we see from the figure, for $q \approx 1$ the system starts in a pure quantum state when the coupling potential is completely disentangled from the two-level atom $\left\{S_P^{(C)}(q;0) = 0\right\}$, but as it evolves the coupling starts to play a role and, after a sequence of few oscillations, the entanglement increases rapidly reaching its maximum value $\{S_{\max} \approx 0.693\}$ and the system reaches a mixed quantum state. This maximum value, characteristic of the maximally correlated system, is obtained when the square root factor in (56) goes to zero and $\lambda_+^{(C)}(q;t) \to \lambda_-^{(C)}(q;t) \to \frac{1}{2}$. For these conditions equation (53) gives the value $S_P^{(C)}(q;t) \to \ln 2 \approx 0.693$. This result is still valid for the bipartite quantum system in general. After reaching the strongest entanglement the coupled system sustains this maximum level for a long time before fluctuations start to appear again. This pattern of short-time oscillations, where the system roughly returns some times to pure quantum states, followed by long time strongest entanglement *plateau* is repeated with a period of $\Delta\tau \approx 60$. The entanglement of the coupled system drastically changes its time behavior for lower values of q. The strongest entanglement plateaus disappear gradually, substituted by a rapid oscillation structure with a well-defined period and a fluctuating amplitude, which are dependent on the value of q. For very low values of the deformation parameter ($q < 0.3$) the resulting oscillations in $S_P^{(C)}(q;t)$ present a pattern resulting from the combination of a small- and a large-amplitude oscillation structures. In this oscillating regime the coupled system

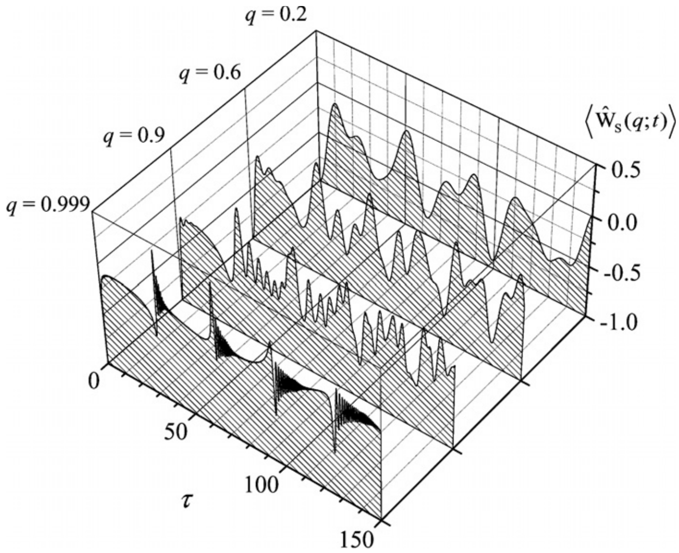

Figure 5. The population inversion factor $\langle \hat{\mathrm{W}}_{\mathrm{S}}(q;t)\rangle$ in terms of the time variable τ calculated for the deformation parameter values $q = 0.2, 0.6, 0.9$ and 0.999.

reaches many times the state of complete disentanglement $\left\{S_{\mathrm{P}}^{(\mathrm{C})}(q;t) = 0\right\}$ and a state of strong entanglement $\left\{S_{\mathrm{P}}^{(\mathrm{C})}(q;t) \approx 0.65 S_{\max}\right\}$.

When we compare figures 2 and 4 it is easy to recognize some resemblance in the pattern of the oscillations of the two observables. The oscillations in both are localized for the same duration but the number of them is twice in $S_{\mathrm{P}}^{(\mathrm{C})}(q;t)$ because their period in this observable is around one-half of the period of the oscillations in the inversion population factor. The strongest entanglement plateaus are time coincident with the collapse events in $\langle \hat{\mathrm{W}}_{\mathrm{C}}(q;t)\rangle$.

Figure 5 is the version of figure 2 when the quantum deformed coupling potential system is assumed in a purely squeezed state at $t = 0$. In the calculations for this initial condition we used the set of deformation parameter values $q = 0.2, 0.6, 0.9$ and 0.999 because, unlike the purely coherent state initial case, here the behavior of the observables changes drastically when $q \approx 1$. Salient features of $\langle \hat{\mathrm{W}}_{\mathrm{S}}(q;t)\rangle$ are: (i) the high number of oscillations in the revival events and the imperfect collapse events (the inversion population factor presents a slow variation in time where it should be null) when $q = 0.999$; (ii) the vanishing of the collapse events and the emergence of Rabi oscillations with frequencies decreasing with the value of the deformation parameter when $q < 0.9$; (iii) the weak excitation of two-level atom when $q < 0.9$ (in general we found that $\langle \hat{\mathrm{W}}_{\mathrm{S}}(q;t)\rangle < 0.0$).

In figure 6, we show the coupling potential partial entropy $S_{\mathrm{P}}^{(\mathrm{S})}(q;t)$ in terms of the time variable $\tau = 2gt$ for the same set of values of the deformation parameter q used in figure 5. If we compare this figure with figure 5 the verification of the resemblance in the oscillation pattern of the two observables is immediate. The reason is the null value of the product $\langle \mathcal{C}_{\mathrm{S}}^{(q)}(t)|\mathcal{D}_{\mathrm{S}}^{(q)}(t)\rangle$, which makes the eigenvalues $\lambda_{\pm}^{(\mathrm{X})}(q;t)$ functions only of $\langle \hat{\mathrm{W}}_{\mathrm{S}}(q;t)\rangle$ (see equations (56) and (57)). For all values of q, we observe that, starting with the initial disentangled state, the coupled system evolves rapidly to a strongly entangled state. When $q \approx 1$, the deviations of the maximum entanglement regime $\left\{S_{\mathrm{P}}^{(\mathrm{S})}(q;t) = S_{\max}\right\}$ are restricted only to depression regions with oscillation packets temporally coincident with the arrival events in $\langle \hat{\mathrm{W}}_{\mathrm{S}}(q;t)\rangle$. As the value of q decreases, the oscillations in $S_{\mathrm{P}}^{(\mathrm{S})}(q;t)$ from the

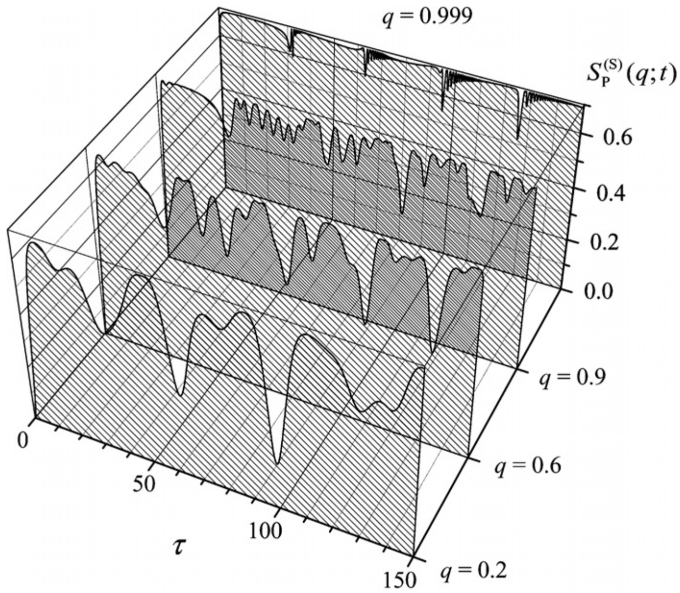

Figure 6. Time evolution of the coupling potential partial entropy $S_P^{(S)}(q;t)$, calculated for the deformation parameter values $q = 0.2$, 0.6, 0.9 and 0.999.

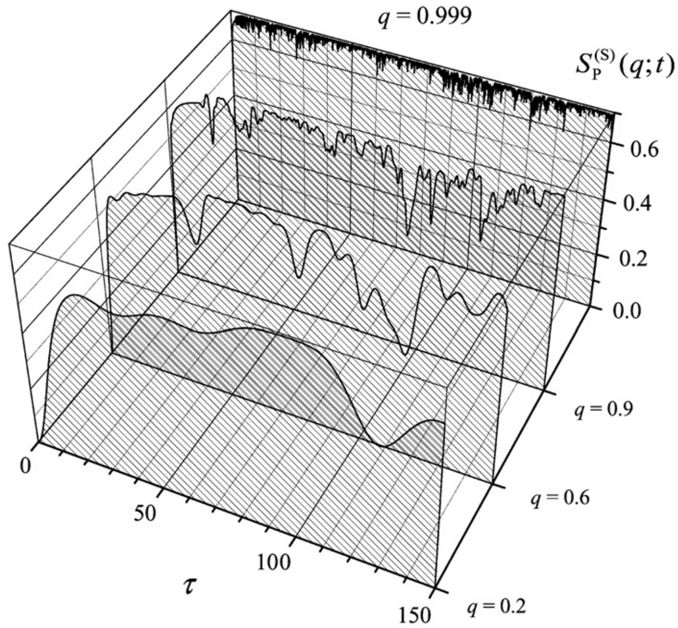

Figure 7. Same as figure 6 for an intensity-dependent interaction Hamiltonian.

maximum entanglement plateau increase their period and present a pattern of superposition of the neighboring oscillation packets. This superposition creates the appearance of progressive dips with increasing depth in the time. Therefore, after some time it should be possible to observe the coupled system returning, for a very short time, to a completely disentangled state (a pure state, as like the initial one).

Figure 7 is the version of figure 6 for an intensity-dependent interaction Hamiltonian. In this case the relevant aspects about the behavior of $S_{\mathrm{P}}^{(\mathrm{S})}(q;t)$ to note are: (i) the behavior becomes increasingly erratic when $q \to 1$ with a large number of narrow and irregular fluctuations in the strongest entanglement plateau; (ii) as the value of q decreases the erratic behavior of $S_{\mathrm{P}}^{(\mathrm{S})}(q;t)$ decreases and the amplitude of the oscillations showing weaker entanglement gets more pronounced, making it possible for the coupled system to come back to its disentangled initial quantum state after a long time. A careful observation of the behavior of $S_{\mathrm{P}}^{(\mathrm{S})}(q;t)$ of the different values of q makes it possible to recognize some resemblance among the results when we consider that their time scale is amplified with the reduction of the q-value. Finally, comparison between figures 6 and 7 shows that the intensity-dependent interaction makes the entanglement of the coupled system more complicated and without a regular pattern when $q \geqslant 0.9$. However, in this case, we do not observe a visible change in the time average of the entanglement of the coupled system when calculated with the two models. For lower values of q the tendency changes. The entropy $S_{\mathrm{P}}^{(\mathrm{S})}(q;t)$ for an intensity-dependent interaction presents a reduction in its fluctuations and, after going rapidly to a strong value, the coupled system entanglement keeps this regime for a long time.

6. Conclusions

In this paper, within a supersymmetric approach, we have studied the system of a two-level atom or molecule interacting with a quantum deformed shape-invariant potential. This exactly soluble and fully quantum-mechanical coupled-channel model may find applications in molecular, atomic and nuclear physics. Taking into account two possible forms of coupling interaction (usual and intensity-dependent interaction models), we studied the quantum dynamics of the coupled system by also considering two possible initial quantum states of the q-deformed shape-invariant potential system [a purely coherent ($\mathrm{X} = \mathrm{C}$) and a purely squeezed ($\mathrm{X} = \mathrm{S}$) states]. We obtained generalized expressions which give the temporal behavior of the atomic population inversion factor $\langle \hat{\mathrm{W}}_{\mathrm{X}}(q;t) \rangle$ as well of the coupling potential partial entropy $S_{\mathrm{P}}^{(\mathrm{X})}(q;t)$. We study the behavior of the expressions for these dynamical variables for a quantum-deformed Pöschl–Teller coupling potential considering different values of the deformation parameter q. The results obtained show how strongly the quantum dynamics of the atomic excitation and the entanglement of the bipartite coupled system depend on the quantum deformation nature of the shape-invariant coupling potential. In our application we used a very simple form for the generalizing functional $\mathcal{Z}_s^{(q)}$ related to the initial state of the deformed coupling potential system. However our formalism is generalized enough to permit the use of many other expressions. One should emphasize that the study of quantum-deformed systems other than the harmonic oscillator is very recent and coupled systems involving these potential systems are mostly unexplored.

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